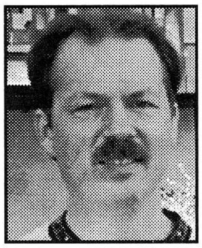
▲ Tangrams Are Not Toys

CORNELIS DE GROOT



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It is possible to assess much of peoples' thinking while observing their struggle to put a puzzle together. Ask anyone to put the seven Tangram pieces back together to form a square, and you will find her or him using trial and error strategies even that person admits to having done this before; adults and children alike. Although people have solved the Tangram puzzle, it is perplexing that each time they encounter the puzzle, it appears just as problematic as before. Each semester, I give this assignment to my pre-service elementary teachers and this ritual consistently repeats itself. Why does this phenomenon happen?

In this article, I describe paper-folding techniques that assist learners in discovering important underlying structures in the Tangram puzzle. The main ideas that I put forward are well suited for upper elementary and middle school teachers and maybe of interest to others as well. I will make the point that knowing these underlying structures enable learners to reconstruct the square from the seven discrete pieces, using mathematical reasoning and strategies. It is more effective to comprehend a structure than it is to remember the actual puzzle.

Manipulatives are occasionally characterized as toys, and sometimes starting as early as second or third grade, students are told that manipulatives, such as cubes or counting with fingers, are for little children. This negative characterization of manipulatives implicates that toys are not appropriate learning tools and that play or manipulation wastes time in the learning of mathematics. I think that this stance is problematic if we believe that the development of learning mathematics is conceptually facilitated by the development of insight into mathematical structures and mathematical relationships. It has been well documented that concrete experiences are needed for many children to accomplish the transition to more abstract mathematical knowledge (for example, Campbell & Johnson, 1995; Cobb, Wood, Yackel, & McNeal, 1993; and Thompson & Lambdin, 1994).

Relationships Among the Pieces

To begin the discovery of the underlying structures of the Tangram, I recommend that learners first develop a good sense of how the seven pieces (two small triangles, one medium triangle, two large triangles, a square, and a parallelogram) are related to each other. There are many activities in books or on the Internet that can assist with helping students form such relationships. You can find a selection of resources at the end of this article. It is particularly important that students realize that all the pieces can be constructed from the small triangle; that is, the medium triangle, the square, and the parallelogram are all composed of two small triangles and the large triangle is made from four small triangles. In figure 1, you can see that the seven pieces together are equivalent to 16 small triangles. This allows the learner to discern the part of the Tangram square that is represented by each piece. For example, the large triangles are each four sixteenths or one fourth of the Tangram square. The medium triangle, the small square, and the parallelogram are each two sixteenths or one eighth of the Tangram square, which leads to the making of a Tangram using these relationships and spatial and logical reasoning. This construction can be demonstrated using a square piece of paper.

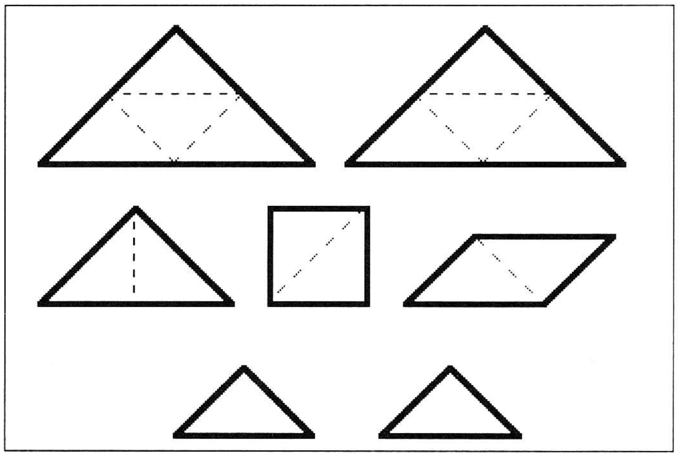

Figure 1. Seven Tangram Pieces are equivalent to 16 small triangles.

Unfolding the Tangram Structure

In this folding method, we will fold along or perpendicular to the diagonals of the square. Others have used a rectangular grid to reveal the structure of Tangrams (see, for example, the University of Exeter Centre for Innovation in Mathematics Teaching web site), but this does not lend itself well to working with the triangular shapes and the parallelogram. As a teacher, you can choose to pre-cut the Tangram squares, but I suggest that you use the diagonal folding methods, so often used in folding paper airplanes. This method might assist the students in viewing triangular pieces that fold on top of each other as congruent (see figure 2).

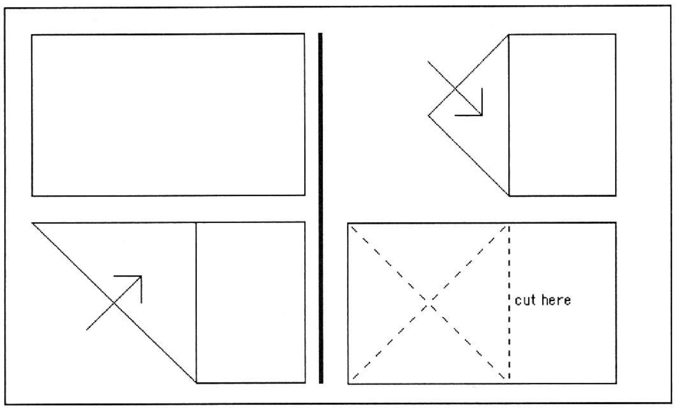

Figure 2. Making a square by diagonal folding.

Upon completion of the construction of the Tangram square, you will be able to identify four triangular regions, each representing one fourth of the entire square (see figure 3a). The two large triangles must be positioned in two of these regions. This is a good moment to investigate how many ways the two large triangles can be positioned in the Tangram square. It appears that there are only two possibilities: adjacent to or opposite of each other (see figure 3b). Determining the number of ways you can place given shapes is a very important step, since it will lead to an efficient strategy in re-construction of the puzzle later on.

Next, fold each vertex of the square to its centre and unfold it (see figure 3c). In this way, a smaller square has been formed, connecting all the midpoints of the original square. It is important to stop here for a moment and examine the size of this inner square. Students may see that the inner square is half the size of the Tangram square. They might also conclude that the two large triangles fit inside this square (see figure 3d). While this is true, it must be discussed that this strategy is not fruitful, since it will be impossible to place the square and the parallelogram in this construction. What can be discovered next? The medium triangle piece can be put in the space on each of the four corners and subsequently be flipped over to the inner square. This construction will show that the Tangram square is made up of eight medium triangles and that the inner square is made of four medium triangles. This in itself would

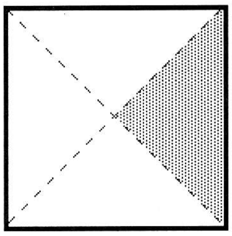

Figure 3a. One fourth of the figure is shaded.

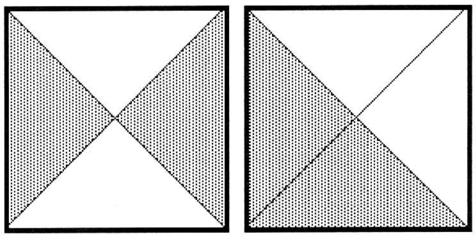

Figure 3b. Ways to place two large triangles.

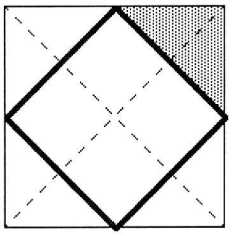

Figure 3c. One eighth of the figure is shaded.

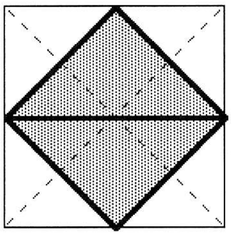

Figure 3d. Placing large triangles here prevents the placement of the square and parallelogram.

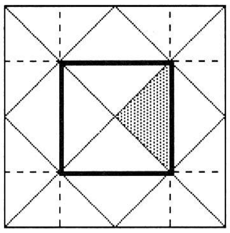

Figure 3e. One sixteenth of the figure is shaded.

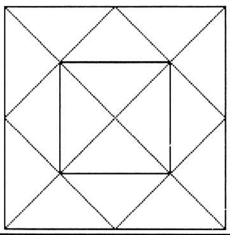

Figure 3f. The fundamental structure for the Tangram.

constitute a visual proof of the previously stated relationship.

Continue folding the vertices back to the centre and again fold the vertices to the centre of the smaller square. I found it was more convenient to fold when I first turned the paper over for this purpose. (You may recognize this construction from your childhood as a fortune telling game. Children often call this a "cootie catcher." It can be folded open and closed with one's thumb and index finger from both hands inserted.) Again, a smaller square is formed. This time, do not immediately unfold it. What is the relationship between this small square, the previous medium square, and the original large square? Unfold the paper completely. Several structures are visible (see figure 3e). First, the three squares that were previously discussed are recognizable. Second, do you see the 16 small triangles? To make this more apparent, have students trace the fold lines that are solid in the figure with a marker, and figure 3f will emerge from this.

We now have the fundamental structure underlying the Tangram. We have already determined where the two large triangles can go and where the medium triangles could fit. First, let us investigate where the square and the parallelogram might fit. The small triangle is the least of our concern, because it can fit in 16 places. You can see that there are only four spots for the square (see figure 4a). Note that due to the symmetry it does not matter which of these four spots is chosen. You can see that the parallelogram must go along the sides of the Tangram square (see figure 4b). There are two possibilities on each side, making a total of eight placements for the parallelogram.

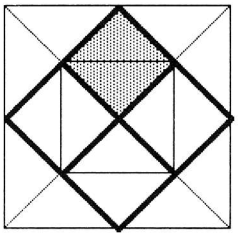

Figure 4a. Placing the square.

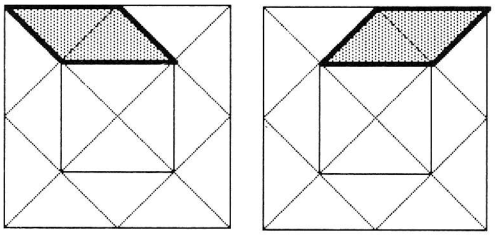

Figure 4b. Placing the parallel-ogram.

Putting It Together

We are now ready to reason backwards to a solution of this puzzle. We will start with the pieces that have the least number of possible placements and work our way up to those with the most: the two large triangles, then the square, then the parallelogram or medium triangle, and finally the two small triangles.

If we start with the two large triangles in opposing positions, we have two places left for the square (see figure 5a). Placing the square in one possibility forces the placement of the parallelogram in the space opposite the square (see figure 5b). There are only two possibilities for this. However, either choice presents a problem; it will be impossible to place the medium triangle anywhere (see figure 5c). We have now excluded the possibility of the two large triangles in opposition. Therefore, they must be adjacent!

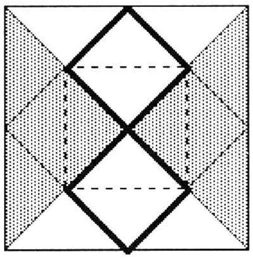

Figure 5a. The two big triangles are placed opposite. Two choices for the square.

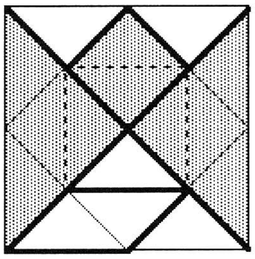

Figure 5b. The parallelogram must go in the space opposite the small square.

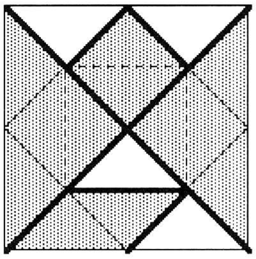

Figure 5c. Placing the medium triangle is impossible.

Start again by placing the two large triangles adjacent to each other (see figure 6a). There are two possibilities for placing the square. Because of the symmetry, it doesn't matter where it is placed (see figure 6b). In turn, this leaves only two possibilities for the parallelogram on the remaining side. One possible placement of the parallelogram will make it impossible to place the

medium triangle, but the other allows the placement of all the remaining pieces (see figures 6c and 6d). Now the puzzle is solved once and for all.

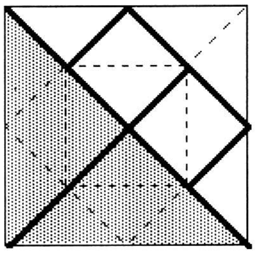

Figure 6a. Large triangles are adjacent and the small square can go in two places.

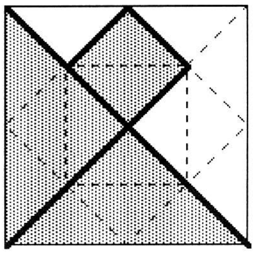

Figure 6b. By symmetry, the square can be arbitrarily placed in one of the two options.

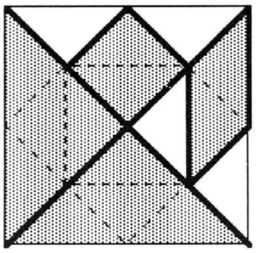

Figure 6c. Only two places for the parallelogram. This option prohibits the placement of the medium triangle.

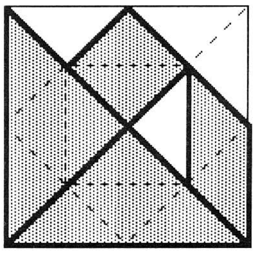

Figure 6d. The puzzle is solved.

The Payoff

We have reached a point where we can reconstruct the Tangram puzzle at any time using the underlying structure, the relationships among the pieces, and logical reasoning. This addresses a large number of important objectives all at once, making it very worthwhile to spend two or three lessons investigating Tangrams. The big payoff is that these strategies model particular mathematical ways of thinking and problem solving, allowing for a transition toward recognizing and describing the underlying structure, which in turn becomes embedded as the mental image for the Tangram. Using such a structure helps students to be successful at solving figures made with the Tangram pieces, as well as deciding whether or not it is possible to construct certain geometric shapes with all seven pieces or a subset of these. I have included a list of additional references for those who are interested in Tangrams and other mathematical puzzles or games. Enjoy!

References

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Additional Resources

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University of Exeter Centre for Innovation in Mathematics Teaching web site: http://www.ex.ac.uk/cimt.

Math Forum at Swarthmore College web site, with an excellent list of links and resources: http://forum.swarthmore.edu/trscavo/tangrams/contents.html. ▲

"Mathematics reveals its secrets only to those who approach it with pure love, for its own beauty."
(Archimedes)